TOPIC —

Problem based On Integration by Method OF Substitution

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Standard integral :-

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$

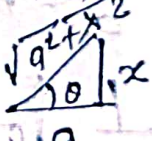
Let $I = \int \frac{dx}{x^2+a^2}$

put $x = a \tan\theta$

$\Rightarrow \tan\theta = \frac{x}{a}$

$\Rightarrow \theta = \tan^{-1} x/a$

Also $x^2 + a^2 = a^2 \sec^2\theta$.



$\sec\theta = \frac{\sqrt{a^2+x^2}}{a} \Rightarrow \sec^2\theta = \frac{a^2+x^2}{a^2}$

Also $\because x = a \tan\theta$

$\Rightarrow dx = a \sec^2\theta \, d\theta$

$\therefore I = \int \frac{a \sec^2\theta \, d\theta}{a^2 \sec^2\theta}$

$= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + C$

$= \frac{1}{a} \cdot \tan^{-1} x/a$

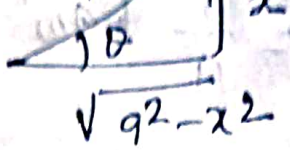
2. $\int \frac{dx}{a^2 - x^2}$

put $x = a\sin\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

$a^2 - x^2 = a^2\cos^2\theta$.

$\Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}$



$= \int \frac{a\cos\theta\, d\theta}{a^2\cos^2\theta}$

$= \frac{1}{a}\int \sec\theta\, d\theta$

$= \frac{1}{a}\log[\sec\theta + \tan\theta]$

$= \frac{1}{a}\log\left[\frac{a}{\sqrt{a^2-x^2}} + \frac{x}{\sqrt{a^2-x^2}}\right]$

$= \frac{1}{a}\log\frac{a+x}{\sqrt{a^2-x^2}}$

$= \frac{1}{a}\log\sqrt{\frac{a+x}{a-x}}$ $\left[\text{use } \log\sqrt{x} = \frac{1}{2}\log x\right]$

$= \frac{1}{a}\cdot\frac{1}{2}\log\frac{a+x}{a-x}$

$= \frac{1}{2a}\log\frac{a+x}{a-x}$

$$I = \int \frac{dx}{x^2 - a^2}$$

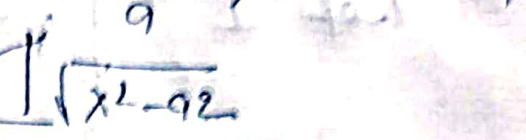
Put $x = a\sec\theta$

$\therefore dx = a\sec\theta\tan\theta\, d\theta$

$x^2 - a^2 = a^2\tan^2\theta$

$\sec\theta = \frac{x}{a} \Rightarrow \theta = \sec^{-1}\frac{x}{a}$

$$= \int \frac{a\sec\theta\cdot\tan\theta\, d\theta}{a^2\tan^2\theta}$$



$$= \frac{1}{a}\int \frac{\sec\theta}{\tan\theta}\, d\theta$$

$$= \frac{1}{a}\int \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{a}\log[\operatorname{cosec}\theta - \cot\theta]$$

$$= \frac{1}{a}\log\left[\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right]$$

$$= \frac{1}{a}\log\left[\frac{x-a}{\sqrt{x^2-a^2}}\right]$$

$$= \frac{1}{a}\log\sqrt{\frac{x-a}{x+a}}$$

$$= \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right]$$

4. $I = \int \sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a}$

$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$

Let $I = \int \sqrt{x^2+a^2}\,dx$

Put $x = a\sinh\theta$

$dx = a\cosh\theta\,d\theta$

$x^2+a^2 = a^2\cosh^2\theta\,d\theta$.

$\Rightarrow \sqrt{x^2+a^2} = a\cosh\theta$

$= \int a\cosh\theta \cdot a\cosh\theta\,d\theta$

$= a^2\int \cosh^2\theta\,d\theta$ $\quad \because \cosh^2 x - \sinh^2 x = 1$

$\cosh^2\theta = \cosh 2\theta + 1$

$= \frac{a^2}{2}\int(\cosh 2\theta + 1)\,d\theta$

$= \frac{a^2}{2}\int \cosh 2\theta\,d\theta + \frac{a^2}{2}\int d\theta$.

$= \frac{a^2}{2}\frac{\sinh 2\theta}{2} + \frac{a^2}{2}\theta$.

$= \frac{a^2}{2\times 2}\,2\sinh\theta\cosh\theta + \frac{a^2}{2}\theta$

$$= \frac{a^2}{4} 2 \sinh\theta \cosh\theta + \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2} \frac{x}{a}\frac{\sqrt{a^2+x^2}}{a} + \frac{a^2}{2} \sinh^{-1}\frac{x}{a}$$

$$= \frac{a^2}{2} \frac{x\sqrt{x^2+a^2}}{a^2} + \frac{a^2}{2} \log\left[x+\sqrt{x^2+a^2}\right]$$

$$= \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \log\left[x+\sqrt{x^2+a^2}\right]$$

Problem 5 $I = \int \sqrt{x^2-a^2}\, dx$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\left\{\log x+\sqrt{x^2-a^2}\right\}$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \cosh^{-1}\frac{x}{a}$$

Let $I = \int \sqrt{x^2-a^2}\, dx$

put $x = a\cosh\theta \Rightarrow \theta = \cosh^{-1}\frac{x}{a}$

$dx = a\sinh\theta\, d\theta$

$x^2-a^2 = a^2\sinh^2\theta$

$\sqrt{x^2-a^2} = a\sinh\theta$

$$\therefore I = \int a\sinh\theta \, a\sinh\theta \, d\theta$$

$$= a^2 \int \sinh^2\theta \, d\theta$$

$$= \frac{a^2}{2} \int (\cosh 2\theta - 1) \, d\theta$$

$$= \frac{a^2}{2} \frac{\sinh 2\theta}{2} - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2} \cdot \frac{2\sinh\theta\cosh\theta}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{a^2}{2} \cdot \frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{a^2 \, x\sqrt{x^2 - a^2}}{2a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right)$$

$$= \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right)$$